

SOME SURFACES WITH NON-POLYHEDRAL NEF CONES

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ABSTRACT. We study the nef cones of complex smooth projective surfaces and give a sufficient criterion for them to be non-polyhedral. We use this to show that the nef cone of $C \times C$, where C is a complex smooth projective curve of genus at least 2, is not polyhedral.

1. INTRODUCTION

There has been a great deal of interest in understanding the various positive cones of curves and divisors on algebraic varieties. Several cases have been analyzed, including symmetric products of curves in [6], [8], abelian varieties in [1], [3], and holomorphic symplectic varieties in [5], [2]. The main result of this paper is the following theorem.

Theorem 1.1. *If C is a smooth projective curve over \mathbb{C} of genus $g \geq 2$, the nef cone of $C \times C$ is not polyhedral.*

We address the cases when $g < 2$. If C has genus 0, it is isomorphic to \mathbb{P}^1 . The nef cone of $\mathbb{P}^1 \times \mathbb{P}^1$ is rational polyhedral and is equal to

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}.$$

If C is a curve of genus 1 and h is an ample class on C , the nef cone of $C \times C$ is precisely

$$\{\alpha \in N^1(C \times C)_{\mathbb{R}} : (\alpha \cdot \alpha) \geq 0, (\alpha \cdot h) \geq 0\}.$$

In this case, the nef cone is not polyhedral.

In section 2, we prove a sufficient criterion for the nef cone of a surface to be non-polyhedral. In section 3, we use this criterion to prove that the nef cone of $C \times C$ is not polyhedral for C a complex smooth projective curve of genus at least 2.

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2. CRITERION FOR NON-POLYHEDRAL NEF CONES

In this section we prove a sufficient criterion for nef cones to not be polyhedral. We begin by fixing some notation. For any smooth projective variety X , we denote by $N^1(X)$ the free and finitely generated \mathbb{Z} -module of numerical equivalence classes of divisors on X . Let $\rho(X)$ be its rank. We use \equiv to denote numerical equivalence. Let $N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The closed convex cone generated by numerical classes of nef divisors is the *nef cone*, denoted by $\text{Nef}(X)$. The closed convex cone generated by numerical classes of effective divisors is the *pseudoeffective cone* denoted by $\text{Psef}(X)$.

In what follows, we assume that X is a smooth projective surface. For such X , $N^1(X)$ is equipped with the usual intersection form and $\text{Psef}(X)$ is the same as the *Mori cone* (denoted by $\overline{\text{NE}}(X)$) which is the dual of the nef cone under the intersection product. Recall that a cone σ is said to be *polyhedral* if it is the positive span of finitely many

vectors. A theorem of Farkas ([4], Pg. 11) tells us that a cone σ is polyhedral if and only if σ^\vee is polyhedral.

Suppose $\rho(X) \geq 3$ and pick an orthogonal basis $\{h, f_1, \dots, f_{\rho(X)-1}\}$ of $N^1(X)_\mathbb{R}$ such that h is ample, $(h \cdot h) = 1$ and $(f_i \cdot f_i) = -1$ for $1 \leq i \leq \rho(X) - 1$. The existence of such a basis follows from the Hodge index theorem.

Proposition 2.1. *For X as above, if there exist e and f such that,*

- (1) $0 \neq e$ is a boundary class of $\overline{NE}(X)$ such that $(e \cdot e) = 0$,
- (2) $0 \neq f$ is a class in the linear span of $\{f_1, \dots, f_{\rho(X)-1}\}$ such that $(e \cdot f) = 0$ and $(e + \mathbb{R}f) \cap \overline{NE}(X) = \{e\}$,

then $Nef(X)$ is not polyhedral.

Proof. Consider the lines $\ell_1 := \{e + sf : s \in \mathbb{R}\}$ and $\ell_2 := \{te + (1 - t)(h \cdot e)h : t \in \mathbb{R}\}$. These lines are distinct because otherwise $e + sf$ would equal h for some value of s , which is impossible since $(e + \mathbb{R}f) \cap \overline{NE}(X) = \{e\}$. The affine 2-plane P spanned by ℓ_1 and ℓ_2 is contained in the affine hyperplane

$$H := \{v \in N^1(X) : (v \cdot h) = (e \cdot h)\}.$$

Since $0 \neq e \in \overline{NE}(X)$, we know that $(e \cdot h) > 0$ by Kleiman's criterion. The image of $\overline{NE}(X) \setminus \{0\}$ in $\mathbb{P}(N^1(X)_\mathbb{R})$ is closed hence compact. Since H maps homeomorphically onto its image in $\mathbb{P}(N^1(X)_\mathbb{R})$, we conclude that $H \cap \overline{NE}(X)$ is compact. It follows that $P \cap \overline{NE}(X)$ is compact, being a closed subset of $H \cap \overline{NE}(X)$.

Assume that $\overline{NE}(X)$ is a polyhedral cone. It follows that $P \cap \overline{NE}(X)$ must be a convex polygon. Since ℓ_1 intersects this convex polygon at precisely one point, e must be a vertex. The class $h' := (e \cdot h)h$ lies in the interior of this polygon, being an ample class. Since $(e + \mathbb{R}f) \cap \overline{NE}(X) = \{e\}$, neither edge of the polygon emanating from e is contained in $(e + \mathbb{R}f)$. Hence, $(h' + \mathbb{R}f)$ is not parallel to either of these edges and it must intersect both edges at precisely one point each, say $h' + \chi_i f$ for $i = 1, 2$. Picking $m > \max(|\chi_1|, |\chi_2|)$ we see that the segment ℓ_3 joining e and $h' + mf$ lies entirely outside $\overline{NE}(X)$, aside from e . A general point on this segment is

$$P_t := te + (1 - t)(h' + mf) \text{ for } 0 \leq t \leq 1.$$

We compute the self-intersection

$$\begin{aligned} (P_t \cdot P_t) &= t^2(e \cdot e) + (1 - t)^2(h' \cdot h') + (1 - t)^2 m^2(f \cdot f) + 2t(1 - t)(e \cdot h') \\ &= (1 - t)((1 - t)(h' \cdot h') + (1 - t)m^2(f \cdot f) + 2t(e \cdot h')) \end{aligned}$$

The term $(1 - t)(h' \cdot h') + (1 - t)m^2(f \cdot f) + 2t(e \cdot h')$ is positive for $t = 1$ since $(e \cdot h') > 0$ because e is pseudoeffective and h' is ample. Hence for t slightly less than 1, this term is positive forcing $(P_t \cdot P_t)$ to be positive. Now this implies that either P_t or $-P_t$ is big. Since $(P_t \cdot h) = (e \cdot h) > 0$, it follows that P_t is big and contained in the interior of $\overline{NE}(X)$, a contradiction! We thus conclude that $\overline{NE}(X)$ is not polyhedral, hence $Nef(X)$ is not polyhedral as well. \square

3. NEF CONE OF $C \times C$

For the remainder of this paper, we focus on a fixed complex smooth projective curve C of genus $g \geq 2$. Let $\Delta \subset C \times C$ be the diagonal and let J be the Jacobian of C . Let

$p_1, p_2 : C \times C \rightarrow C$ be the projection morphisms. Let e_i be the numerical class of a fiber of p_i and $\delta := \Delta - e_1 - e_2$. Recall (see [7], Section 1.5) that

$$(1) \quad (e_1 \cdot e_1) = (e_2 \cdot e_2) = (e_1 \cdot \delta) = (e_2 \cdot \delta) = 0, \quad (e_1 \cdot e_2) = 1, \quad \text{and} \quad (\delta \cdot \delta) = -2g.$$

Furthermore, we have

$$\begin{aligned} N^1(C \times C) &= p_1^* N^1(C) \oplus p_2^* N^1(C) \oplus \text{Hom}(J, J) \\ &= \mathbb{Z} \oplus \mathbb{Z} \oplus \text{Hom}(J, J). \end{aligned}$$

Since $\text{rank}_{\mathbb{Z}}(\text{Hom}(J, J)) \geq 1$, it follows that $\rho(C \times C) \geq 3$. It is well known that the Mori cone is a full-dimensional cone in $N^1(C \times C)_{\mathbb{R}}$.

Lemma 3.1. *For $\nu \in \overline{NE}(C \times C)$, we have $(e_2 \cdot \nu) \geq 0$.*

Proof. This is immediate since $e_2 \equiv C \times \{P\}$ and is nef, hence is nonnegative on $\overline{NE}(C \times C)$. \square

We need the following result of Vojta.

Proposition 3.2 (Proposition 1.5, [9]). *Let $Y(r, s) := a_1 e_1 + a_2 e_2 + a_3 \delta$ where $a_1 = \sqrt{\frac{g+s}{r}}$, $a_2 = \sqrt{(g+s)r}$ and $a_3 = \pm 1$, for $r, s \in \mathbb{R}_{>0}$. If*

$$r > \frac{(g+s)(g-1)}{s},$$

then $Y(r, s)$ is nef.

In his paper, Vojta only considers the case $a_3 = 1$. For completeness, we sketch (with suitable modifications) the proof of Proposition 3.2 below.

Proof due to Vojta. Assume, arguing by contradiction, that there exists a curve C_0 (not necessarily smooth) on $C \times C$ such that $(C_0 \cdot Y(r, s)) < 0$. We may assume that C_0 is irreducible. Note that it is not a fiber of p_i for $i = 1, 2$ since $(e_i \cdot Y(r, s)) \geq 0$. Applying the adjunction formula, we get

$$\begin{aligned} (C_0^2) + (2g-2)((C_0 \cdot e_1) + (C_0 \cdot e_2)) &= (C_0^2) + (C_0 \cdot K_{C \times C}) \\ &= 2p_a(C_0) - 2 \\ &\geq 2p_g(C_0) - 2 \\ &\geq (2g-2)(C_0 \cdot e_1), \end{aligned}$$

where $p_a(C_0)$ and $p_g(C_0)$ are the arithmetic and geometric genera¹ of C_0 . Note that the last inequality follows by applying Riemann-Hurwitz to $p_1 \circ \eta : \widetilde{C}_0 \rightarrow C$, where $\eta : \widetilde{C}_0 \rightarrow C_0$ is the normalization. The composition $p_1 \circ \eta$ is a finite morphism because C_0 is not a fiber of either projection. We can then conclude that

$$(2) \quad (C_0^2) + (2g-2)(C_0 \cdot e_2) \geq 0.$$

Write $C_0 \equiv b_0 \delta + b_1 e_1 + b_2 e_2 + \nu$ where ν is orthogonal to δ, e_1 and e_2 in $N^1(C \times C)_{\mathbb{R}}$. The Hodge index theorem forces $(\nu \cdot \nu) \leq 0$. Using this and (2), we compute

$$2b_1 b_2 + (2g-2)b_1 \geq 2gb_0^2.$$

Since $b_1 \geq 0$ and is an integer (being equal to $(C_0 \cdot e_2)$) we have $b_1^2 \geq b_1$ and can write

$$(3) \quad 2b_1 b_2 + (2g-2)b_1^2 \geq 2gb_0^2.$$

¹Recall that the geometric genus of a singular curve is defined as the genus of its normalization.

Now we apply $(C_0 \cdot Y(r, s)) < 0$ which gives

$$(4) \quad b_1 \sqrt{(g+s)r} + b_2 \sqrt{\frac{g+s}{r}} < 2a_3 g b_0.$$

Since $b_1, b_2 \geq 0$, the left hand side of (4) is nonnegative. Thus we can square (4)² and combine it with (3) to get

$$(g+s)(b_2^2/r + 2b_1b_2 + b_1^2r) < 4g(b_1b_2 + (g-1)b_1^2).$$

Rearranging this, we get

$$b_2^2(g+s)/r + 2b_1b_2(s-g) + b_1^2((g+s)r - 4g(g-1)) < 0.$$

This is a quadratic form in b_1, b_2 and therefore its discriminant must be nonnegative. Solving for r then gives

$$r \leq \frac{(g+s)(g-1)}{s}.$$

However this contradicts the hypothesis about r . Hence no such C_0 can exist and $Y(r, s)$ must be nef. \square

We use Proposition 3.2 to prove the following result.

Proposition 3.3. *If $\nu = e_2 + q\delta$ and $q \neq 0$ then*

$$\nu \notin \overline{NE}(C \times C).$$

Proof. If we pick a_3 so that $a_3q = |q|$, then

$$\begin{aligned} (Y(r, s) \cdot \nu) &= a_1 - 2gqa_3 \\ &= \sqrt{\frac{g+s}{r}} - 2|q|g. \end{aligned}$$

Now letting $s = 1$ and r tend to ∞ , we get that $\sqrt{\frac{g+s}{r}}$ approaches 0. This forces $(Y(r, 1) \cdot \nu)$ to approach $-2|q|g < 0$, implying that for $r \gg 0$, $(Y(r, s) \cdot \nu) < 0$. We conclude that ν is not pseudoeffective, since its intersection with a nef divisor is negative. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to apply Proposition 2.1 with $h = \frac{e_1 + e_2}{2}$, $e = e_2$ and $f = \delta$. Proposition 3.3 tells us that condition (2) in Proposition 2.1 is satisfied. \square

Remark 3.4. Observe that for C/k , where $k = \bar{k}$ is a field of characteristic $p > 0$, Theorem 1.1 is easily seen to be true because the graph of the e^{th} power of Frobenius, denoted by Δ_e , is irreducible and $(\Delta_e \cdot \Delta_e) < 0$. It follows that $\overline{NE}(C \times C)$ has infinitely many extremal rays, hence is not polyhedral.

²This is the only step where a_3 makes an appearance and it is immediately being squared. The proof proceeds exactly as in [9] from here.

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